The c-Cluster Complex and The c-Cambrian Fan

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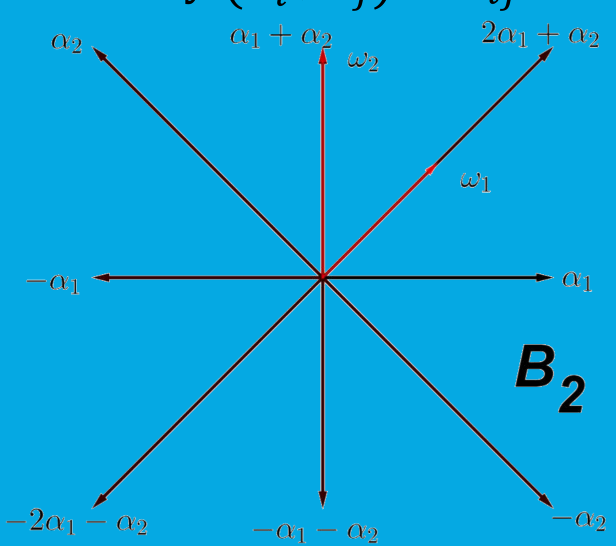
Abstract

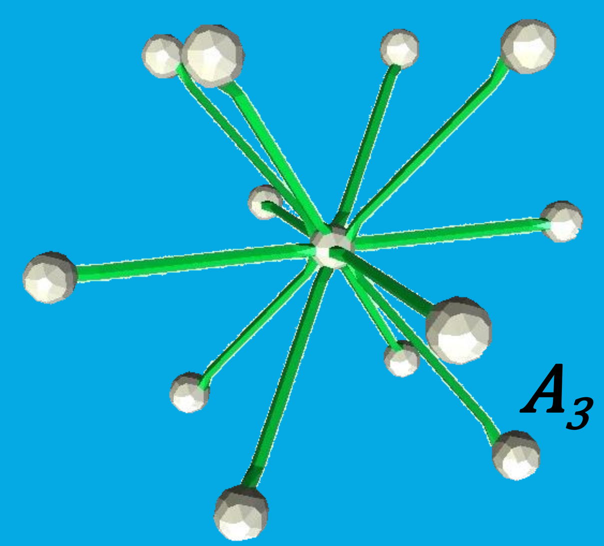
We develop a new approach to the theory of finite crystallographic root systems in order to study the c -cluster complex and the c -Cambrian fan. This new approach stresses a more uniform and cleaner geometric approach over opaque combinatorial arguments. As an application, we vastly simplify the proofs of many known important results, and in the process prove new extensions of these results. This new approach will hopefully prove useful in gaining further insight on major unsolved problems in the theory of cluster algebras, in particular the frieze problem.

Introduction

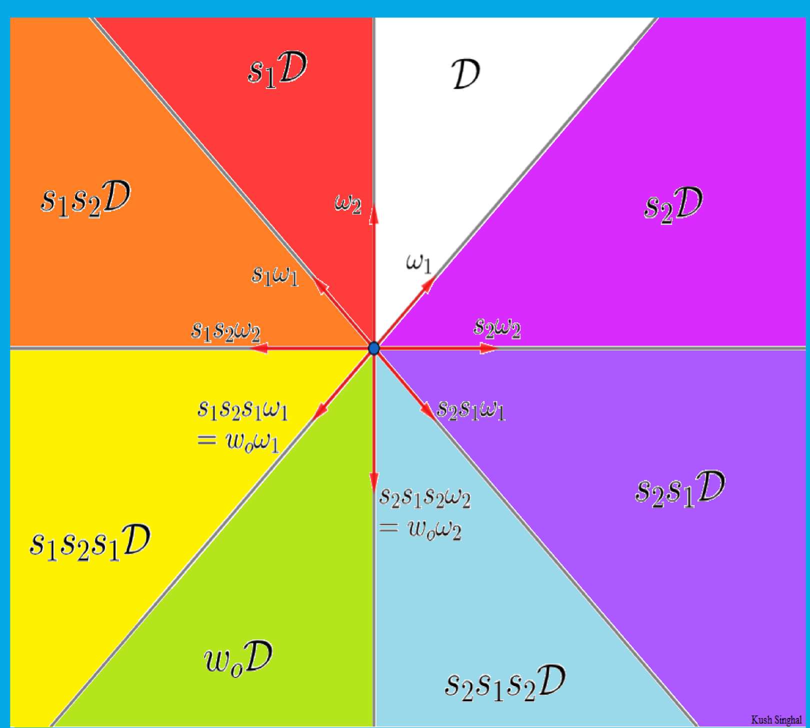
What are Finite Root Systems?

- A *rank r crystallographic finite root system* Φ [1] is a finite set of vectors in an r-dimensional real vector space $\mathfrak{h}^*$ with an inner product $\langle , \rangle$ such that
  - $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ for all $\alpha \in \Phi$;
  - (crystallographic condition) for any $\alpha, \beta \in \Phi$, we have
  $$(\alpha^\vee, \beta) \coloneqq \frac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z};$$
  - for any $\alpha, \beta \in \Phi$, the *reflection* $s_\alpha\beta \coloneqq \beta - (\alpha^\vee, \beta)\alpha \in \Phi$.
- There exists a basis $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ of *simple roots* for $\mathfrak{h}^*$such that every root $\alpha \in \Phi$ can be expressed as $\alpha = \sum_i c_i\alpha_i$ where the c_i are either all non-negative integers, or all non-positive integers.
- Fixing a choice of simple roots, the *fundamental weights* $\underline{\omega}$ are a basis for $\mathfrak{h}^*$ which satisfy $(\alpha_i^\vee, \omega_j) = \delta_{ij}$.





- The *Weyl Group* W is the orthogonal group generated by the reflections s_α. Elements of W can be written as words $s_{j_1}s_{j_2}\cdots s_{j_k}$.
- There is a unique *longest element* w_o of length $|\Phi|$, in the sense that any other element of W can be written as a product of (strictly) less than $|\Phi|$ simple reflections. It is an involution, and for each $i \in [1, r]$
$$w_o\omega_i = -\omega_{i^*}$$
for some $i^* \in [1, r]$.
- The *fundamental chamber* $\mathcal{D}$ is the cone generated by the fundamental weights. The Weyl group acting on $\mathcal{D}$ creates the *Coxeter Fan* of cones.

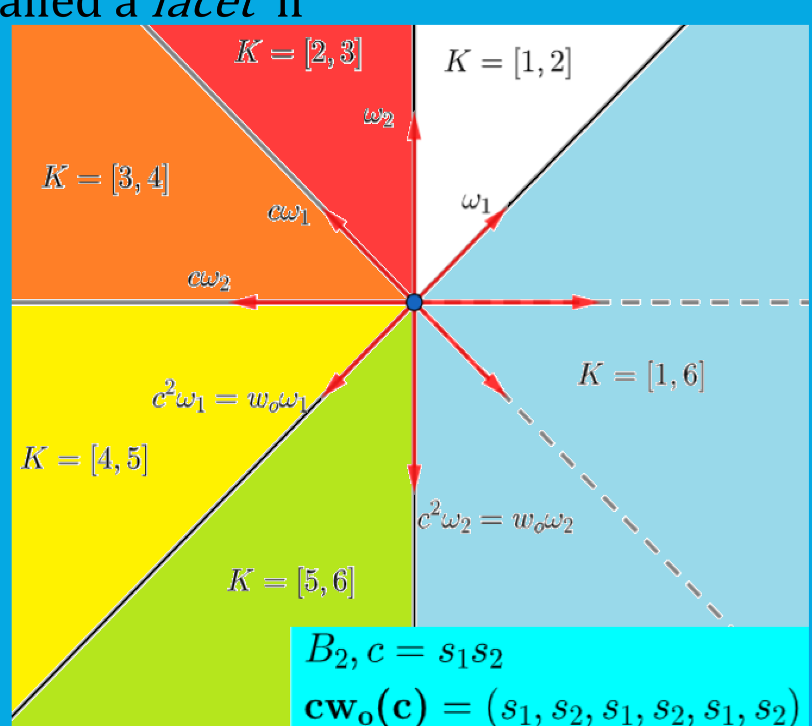


All About Coxeter Elements

- *A Coxeter element* $c = s_{i_1}s_{i_2}\cdots s_{i_r}$ is a product of all simple reflections taken exactly once, in any order (i.e. $(i_1, \dots, i_r)$ is a permutation of $(1,2,\dots,r)$) [1].
- This defines a partial order $\preccurlyeq_c$ on $[1, r]$, where $i \prec_c j$ iff i occurs before j in *every* permutation of $[1, r]$ that gives the same value of c. [2]
- It is known that 1 is not an eigenvalue of c, i.e. $(1 - c): \mathfrak{h}^* \to \mathfrak{h}^*$ is invertible.
- The *fundamental c -roots* are the roots given by
$$\beta_{i;c} \coloneqq (1 - c)\omega_i.$$
- For any root $\alpha \in \Phi$, there exists a unique smallest $b_c(\alpha) \in \mathbb{Z}_{\geq 0}$ and a unique $i_c(\alpha) \in [1, r]$ such that $\alpha = c^{b_c(\alpha)}\beta_{i_c(\alpha);c}$ [2]. Here, $b_c(\alpha)$ is called the *block number*, and $i_c(\alpha)$ the *orbit index*, of α.

The c-Cluster Complex

- Fix a reduced word $\underline{c}$ for the Coxeter element c (i.e. fix a permutation of $[1, r]$). This fixes a reduced word $\boldsymbol{w_o}(\underline{c})$ for w_o, called the *$\underline{c}$-sorting word for w_o*. Let
$$\underline{c}\boldsymbol{w_o}(\underline{c}) = (s_{i_1}, s_{i_2}, \dots, s_{i_d}).$$
- A subset $K = \{k_1 < \cdots < k_r\} \subset [1, d]$ is called a *facet* if
$$s_{i_1}s_{i_2}\cdots\widehat{s_{i_{k_1}}}\cdots\widehat{s_{i_{k_r}}}\cdots s_d = w_o.$$
- Define an abstract simplicial complex structure on $[1, d]$ by taking the collection of all subsets of every facet. This is the *$cw_o(c)$ sub-word complex (or c -cluster complex)* [3].
- For each $k \in [1, d]$, set $\omega_k^{\underline{c}} = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}\omega_{i_k}$. Then, for each facet K, we have the set of r vectors $\{\omega_k^{\underline{c}}: k \in K\}$. It is known that this is a basis for $\mathfrak{h}^*$.



- Let C_K be the cone generated by $\{\omega_k^{\underline{c}}: k \in K\}$. The collection of all cones C_K as K runs through all facets generates the *c-Cambrian Fan* [3].

Results

We fix a Coxeter element c throughout.

Definition: c-Order on Φ.
Given two roots $\alpha, \beta \in \Phi$, we say that $\alpha \leq_c \beta$ iff either
$$b_c(\alpha) < b_c(\beta), \text{ or } b_c(\alpha) = b_c(\beta) \text{ and } i_c(\alpha) \preccurlyeq_c i_c(\beta).$$

Lemma: For any reduced word for $\underline{c}$, if $\beta_k^{\underline{c}} \leq_c \beta_{k'}^{\underline{c}}$, then $k < k'$, where
$$\beta_k^{\underline{c}} \coloneqq s_{i_1}s_{i_2}\cdots s_{i_{k-1}}\alpha_{i_k}$$
with $\underline{c}\boldsymbol{w_o}(\underline{c}) = (s_{i_1}, s_{i_2}, \dots, s_{i_d})$.

Recall the definition $\omega_k^{\underline{c}} = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}\omega_{i_k}$ (given a reduced word). It is known that $\beta_k^{\underline{c}} = (1-c)\omega_k^{\underline{c}}$ for all $k \in [1, d]$.

Theorem 1: Suppose $\beta, \beta' \in \Phi(c)$ are such that either both $\beta, \beta' \in \Phi^+$, or both $c^{-1}\beta, c^{-1}\beta' \in \Phi^+$, and set $\omega \coloneqq (1-c)^{-1}\beta$ and $\omega' \coloneqq (1-c)^{-1}\beta'$. Define
$$p \coloneqq (\beta^\vee, \omega') \quad \text{and} \quad p' \coloneqq (\beta'^\vee, \omega).$$
Then, the following four statements hold:
1) $pp' \leq 0$, i.e. p, p' have opposite sign (or at least one of them is zero).
2) If $\beta <_c \beta'$, then $p \leq 0 \leq p'$.
3) If $(p, p') \neq (0,0)$, then $\beta <_c \beta'$ if and only if $p \leq 0 \leq p'$.
4) We have
$$\langle\beta, \beta\rangle p + \langle\beta', \beta'\rangle p' = 2\langle\beta, \beta'\rangle.$$
Consequently, $(p, p') \neq 0$ if β and β' are not orthogonal.

As examples to showcase our new approach, we provide one new definition, one new result, and one extension of a previously known result.

Lemma-Definition: Given any $B \subset \Phi(c)$, we define the *ordered product*
$$\Pi_c(B) \coloneqq s_{\beta_1}\cdots s_{\beta_n}$$
where $B = \{\beta_1, \dots, \beta_n\}$ is *any* ordering on B such that for any $i < j$, either β_i and β_j are orthogonal, or $\beta_i <_c \beta_j$.

The ordered product allows us to define an abstract simplicial complex structure on $\Phi(c)$, by declaring $B \subset \Phi(c)$ to be a facet iff $|B| = r$ and $\Pi_c(B) = c^{-1}$.

This coincides with the previous notion of a facet, by the correspondence
$$K \subset [1, d] \leftrightarrow B_K \coloneqq \{\beta_k^{\underline{c}}: k \in K\}.$$

- Now, for any $w \in W$, suppose $\underline{w} = (s_{j_1}, \cdots, s_{j_\ell})$, and let $\beta_{t;\underline{w}} \coloneqq s_{j_1}\cdots s_{j_{t-1}}\alpha_{j_t}$. Then,
$$\text{inv}(w) \coloneqq \{\beta_{t;\underline{w}}: t \in [1, \ell]\} = \Phi^+ \cap w\Phi^-$$
is the set of all positive roots sent to negative roots by w^{-1} [1].
- It is known that for any $w \in W$, there exists a unique facet $B \subset \Phi(c)$ such that $w\mathcal{D} \subseteq C_B$. Moreover, given a facet B, there exists a unique $w \in W$ with minimal length satisfying $w\mathcal{D} \subseteq C_B$ [3]. Such an element is called *c-sortable.*

Theorem (Characterization of c-Sortable Elements): An element $w \in W$ is c-sortable iff for each $t, t' \in [1, \ell]$ with $t < t'$, either $\beta_{t;\underline{w}}$ and $\beta_{t';\underline{w}}$ are orthogonal, or $\beta_{t;\underline{w}} <_c \beta_{t';\underline{w}}$, where $\underline{w}$ is the *$\underline{c}$-sorting word* of w. Moreover, if either condition holds, then $\Pi_c(\text{inv}(w)) = w^{-1}$.

Theorem (Sign-Coherence): Suppose $B \subset \Phi(c)$ is a facet, and let C_B be the cone generated by $(1-c)^{-1}\beta$ for $\beta \in B$. Then, (it is known that) for each $i \in [1, r]$, each cone C_B lies completely in one of the two right half spaces
$$H_{\alpha_i}^+ \coloneqq \{\lambda \in \mathfrak{h}^*: (\alpha_i^\vee, \lambda) \geq 0\} \text{ or } H_{\alpha_i}^- \coloneqq \{\lambda \in \mathfrak{h}^*: (\alpha_i^\vee, \lambda) \leq 0\};$$
i.e. either $C_B \subset H_{\alpha_i}^+$ or $C_B \subset H_{\alpha_i}^-$ [4].
Moreover, for fixed $i \in [1, r]$, we have:
- if $C_B \subset H_{\alpha_i}^+$, then $C_B \cap (\mathbb{R}\alpha_i)^\perp$ contains the (possibly trivial) cone spanned by
$$\{(1-c)^{-1}\beta: \beta \in B, \alpha_i \nleq_c \beta\}$$
- if $C_B \subset H_{\alpha_i}^-$, then $C_B \cap (\mathbb{R}\alpha_i)^\perp$ contains the (possibly trivial) cone spanned by
$$\{(1-c)^{-1}\beta: \beta \in B, \alpha_i \ngeq_c \beta\}$$

Significance

- Theorem 1 provides an easy method to compare two roots under the c-order. This also allows us to reduce the proof of many results (both known as well as new) into a direct computation involving inner products.
- The c-order allows us to give more intrinsic definitions for important structures, thus avoiding some of the combinatorial book-keeping.
- Our proofs avoid opaque inductive arguments on reduced words of c.

References

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